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OPERATOR KANTOROVICH TYPE INEQUALITY

MOHAMMAD SAL MOSLEHIAN

ABSTRACT. We present some operator Kantorovich inequalities involving unital positive linear mappings and the operator geometric mean in the setting of semi-inner product C^*-modules. This talk is based on [M.S. Moslehian, Recent developments of the operator Kantorovich inequality, Expo. Math. 30 (2012), no. 4, 376-388].

1. INTRODUCTION

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and I be the identity operator. In the case when $\dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$ and denote its identity by I_n. An operator $A \in \mathbb{B}(\mathscr{H})$ is said to be positive (positive semi-definite for matrices) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and we write $A \geq 0$. For selfadjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $B \geq A$ if $B - A \geq 0$. If $A \geq 0$ is invertible, we call it strictly positive (positive-definite for matrices) and write $A > 0$.

In 1948, Leonid Vital'evich Kantorovich [18] introduced the following inequality

$$\langle Hx, x\rangle\langle H^{-1}x, x\rangle \leq (\lambda_1 + \lambda_n)^2/4\lambda_1\lambda_n \tag{1.1}$$

where $x = (x_1, \cdots, x_n)$ is a unit vector in $\mathbb{C}^n$ and H is an $n \times n$ positive-definite matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Using the spectral decomposition $A = U^*\mathrm{diag}(\lambda_1, \cdots, \lambda_n)U$, we see that inequality (1.1) reduces to

$$\left(\sum_{j=1} \lambda_j |x_j|^2\right)\left(\sum_{j=1}^{n} \frac{1}{\lambda_j}|x_j|^2\right) \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}, \tag{1.2}$$

which can be proved by utilizing the arithmetic mean–geometric mean inequality. Of course, the Kantorovich inequality is still valid for an operator A acting on an

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infinite dimensional Hilbert space $\mathscr{H}$ with $0 < m \leq A \leq M$ as follows:

$$\langle Ax, x\rangle\langle A^{-1}x, x\rangle \leq \frac{(M+m)^2}{4mM} \quad (x \in \mathscr{H}, \| x \|= 1)\,.$$

Replacing x by $A^{1/2}x/ \parallel A^{1/2}x \parallel$ in the above inequality, we get the following equivalent form of the Kantorovich inequality:

$$\langle A^2x, x\rangle \leq \frac{(M+m)^2}{4mM}\langle Ax, x\rangle^2 \quad (x \in \mathscr{H}, \| x \|= 1)\,. \tag{1.3}$$

As reported in a survey presented in 1997 by Watson, Alpargu and Styan [36], inequality (1.2) was established five years earlier in 1943 by Roberto Frucht [16] (it was originally due to Charles Hermite); see also [25]. It is noticed in [36] that the Kantorovich inequality is equivalent to five other inequalities due to Schweitzer (1914), Pólya and Szegö (1925), Cassels (1951), Krasnosel'skiĭ and Kreĭn (1952), and Greub and Rheinboldt (1959). The Kantorovich inequality is a useful tool in numerical analysis and statistics for establishing the rate of convergence of the method of steepest descent. During the past decades several formulations, extensions or refinements of the Kantorovich inequality in various settings have been introduced by many mathematicians; see [8, 15] and references therein.

The first generalization of the Kantorovich inequality is due to Greub and Rheinboldt. In 1959, they [17] showed that if $A \in \mathbb{B}(\mathscr{H})$ such that $0 < mI \leq A \leq MI$, then

$$\langle x, x\rangle^2 \leq \langle Ax, x\rangle\langle A^{-1}x, x\rangle \leq \langle x, x\rangle^2(M+m)^2/(4Mm) \tag{1.4}$$

for any $x \in \mathscr{H}$. They also proved that their inequality (1.4) is equivalent to

$$\langle Ax, Ax\rangle\langle Bx, Bx\rangle \leq \langle Ax, Bx\rangle^2(MM' + mm')^2/(4mm'MM')\,,$$

when B is a selfadjoint operator permutable with A, $x \in \mathscr{H}$ and $0 < m'I \leq B \leq M'I$.

In the next year Strang [35] generalized inequality (1.4) by showing that if $T \in \mathbb{B}(\mathscr{H})$ is invertible, $\| T \|= M$ and $\| T^{-1} \|= m^{-1}$, then

$$|\langle Tx, x\rangle\langle y, T^{-1}y\rangle)| \leq \frac{(M+m)^2}{4mM}\langle x, x\rangle\langle y, y\rangle\,,$$

for all $x, y \in \mathscr{H}$ and that the bound is the best possible.

Ky Fan [9] in 1966 generalized the inequality above by showing that if $0 < mI \le H \le MI$ and $x_1, \cdots, x_m$ are vectors in $\mathbb{C}^n$ such that $\sum_{i=1}^n \| x_i \|^2 = 1$, then

$$\sum_{j=1}^{m} \langle H^p x_j, x_j\rangle \left[\sum_{j=1}^{m} \langle H x_j, x_j\rangle\right]^{-p} \le (p-1)^{p-1} p^{-p} \left(b^p - a^p\right)^p \left((b-a)(ab^p - ba^p)^{p-1}\right)^{-1}$$

herein p is any integer different from 0 and 1. In 1997 Mond and Pečarić [28] gave an operator version of Ky Fan's inequality.

Of course, there exist some integral and discrete versions of the Kantorovich inequality in the literature. For instance, if $0 \le m \le f \le M$, then

$$\int_E f^2 d\mu \le \frac{(m+M)^2}{4mM} \left(\int_E f d\mu\right)^2,$$

which is the additive version of the Grüss type inequality $\int_E f^2 d\mu - \left(\int_E f d\mu\right)^2 \le (M-m)^2/4$. In 1988 Andrica and Badea [3] stated a Grüss inequality for positive linear functionals and applied it to get a Kantorovich inequality; see also [31].

Around the year 1993, Mond and Pečarić [26] obtained several kinds of extensions of the Kantorovich inequality. They proved that if Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$ and $A \in \mathbb{B}(\mathscr{H})$ is a positive operator satisfying $0 < mI \le A \le MI$, then

$$\Phi(A^{-1}) \le \frac{(m+M)^2}{4mM} \Phi(A)^{-1} \quad \& \quad \Phi(A) - \Phi(A^{-1})^{-1} \le (\sqrt{M} - \sqrt{m})^2 I.$$

The second one is called an additive version of the Kantorovich inequality.

If $A_1, \cdots, A_k \in \mathbb{M}_n$ are positive-definite matrices with eigenvalues contained in the interval $[m, M] \subseteq (0, \infty)$, Mond and Pečarić [27] in 1994 proved that

$$\sum_{j=1}^{k} U_j A_j^{-1} U_j^* \le \frac{(m+M)^2}{4mM} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{-1}, \tag{1.5}$$

where $U_1, \cdots, U_k$ are $m \times n$ matrices such that $\sum_{j=1}^k U_j U_j^* = I_m$, which is a generalization of a result of Marshal and Olkin [24]. The result of Mond and Pečarić [27], in turn, was generalized by Spain [34] in 1996.

In 1996, using the operator geometric mean, Nakamoto and Nakamura [32] proved that

$$\Phi(A) \# \Phi(A^{-1}) \le \frac{M+m}{2\sqrt{Mm}}, \tag{1.6}$$

whenever $0 < m \le A \le M$ and Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$.

A discussion of order-preserving properties of increasing functions through the Kantorovich inequality is presented by Fujii, Izumino, Nakamoto and Seo [12] in 1997. They showed that if $A, B > 0, A \geq B$ and $0 < m \leq A \leq M$, then

$$\frac{(m+M)^2}{4mM} A^2 \geq B^2 .$$

They also proved that the Kantorovich inequality is equivalent to the following noncommutative variant of the GreubRheinboldt inequality

$$\langle Ax, x\rangle\langle Bx, x\rangle \leq \frac{(m+M)^2}{4mM} \langle A\sharp Bx, x\rangle^2 ,$$

in which A, B are positive operators satisfying $0 < m \leq A, B \leq M$ and x is an arbitrary vector. In 2006 Yamazaki [37] generalized the inequality above to n-operators via the geometric mean introduced by Ando–Li–Mathias [2].

Some other extensions of the Kantorovich inequality were given by Furuta [14] in the year 1998. He proved that if A, B are positive operators, $A \geq B > 0$ and $MI \geq B \geq mI > 0$, then

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} A^p \geq B^p$$

holds for all $p \geq 1$. The constant $\kappa_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}$ is called the Ky Fan–Furuta constant in the literature; see [15, 13].

In 1998 Kitamura and Seo [19] established a Kantorovich inequality involving the Hadamard product. They proved that if A is a positive operator such that $0 < m \leq A \leq M$, then

$$(A^2 \circ I)^{1/2}(A^{-2} \circ I)^{1/2} \leq \frac{M^2 + m^2}{2Mm}$$

in which $\circ$ denotes the Hadamard product defined for an arbitrary orthonormal basis $\{e_n\}$ of H by $\langle (A \circ B)e_i, e_j\rangle = \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle$. Of course, some Kantorovich type inequalities involving the Hadamard product of matrices have already obtained by Liu and Neudecker [22].

In the next year Yamazaki and Yanagida [38] characterized the chaotic order in terms of the Kantorovich inequality by showing that $\log A \geq \log B$ if and only if $\frac{(m^p+M^p)^2}{4m^pM^p} A^p \geq B^p$ for all $p \geq 0$.

Given a positive-definite matrix H with eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$ the Wielandt inequality states that

$$\frac{|\langle x, Hy\rangle|^2}{\langle x, Hx\rangle\langle y, Hy\rangle} \leq \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2,$$

where $\{x, y\}$ is an orthonormal set. In 2001, Zhang [39] proved that the Wielandt and Kantorovich inequalities are equivalent.

In 2001, T. Ando [1] presented some Kantorovich-type inequalities as upper estimates of the maximum spectra of $\Phi(A)^{-1}\Phi(A^2)\Phi(A)^{-1}$, where Φ is a linear map on a C^*-algebra $\mathscr{A}$ and $A \in \mathscr{A}$. Five years later, inspired by the paper of Ando [1], Li and Mathias [21] established a weak majorization inequality for singular values extending the Kantorovich inequality .

In 2006 the authors of [10] established another noncommutative Kantorovich inequality. They proved that if A, B are positive operators such that $0 < mI \leq A, B \leq MI$, then

$$\frac{2\sqrt{Mm}}{M+m}\left(\frac{A+B}{2}\right) \leq A\sharp B \leq \frac{M+m}{2\sqrt{Mm}}\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

Also, Bourin [5] showed, among several Kantorovich type inequalities, that if H is a positive-definite matrix such that $0 < m \leq H \leq M$, then

$$\| Hx \| \leq \frac{m+M}{2\sqrt{mM}}\langle x, Hx\rangle .$$

Around the year 2008, Dragomir [7] gave several Kantorovich type inequalities involving norms and numerical radii for operators acting on a Hilbert space.

In 2010 Niezgoda [33] obtained some Kantorovich type inequalities for ordered linear spaces.

In 2011, the authors of [30] presented a Diaz–Metcalf type operator inequality and applied it to get a unified approach to several operator inequalities including the Pólya–Szegö, Greub–Rheinboldt, Kantorovich, Shisha–Mond, Schweitzer, Cassels and Klamkin–McLenaghan inequalities.

The notion of semi-inner product C^*-module is a generalization of that of semi-inner product space in which the semi-inner product takes its values in a C^*-algebra instead of the field of complex numbers. We can define a semi-norm on a semi-inner product $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ over a C^*-algebra $\mathscr{A}$ by $\| x \| = \| \langle x, x\rangle \|^{\frac{1}{2}}$, where the latter norm denotes that of $\mathscr{A}$. A *pre-Hilbert $\mathscr{A}$-module* (or an *inner-product module*) is a semi-inner product module in which $\|\cdot\|$ defined as above is a norm. If this norm is complete then $\mathscr{X}$ is called a *Hilbert C^*-module.* Each C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert $\mathscr{A}$-module via $\langle a,b\rangle = a^*b$ $(a,b \in \mathscr{A})$. When $\mathscr{X}$ is a Hilbert C^*-module, we denote by $\mathbb{B}(\mathscr{X})$ the C^*-algebra of all adjointable operators on $\mathscr{X}$. For every $x \in \mathscr{X}$ the absolute value of x is defined by $|x| = \langle x,x\rangle^{\frac{1}{2}} \in \mathscr{A}$. Some standard references for C^*-modules are [20, 23].

Using the polar decomposition, the authors of [11] obtained a new Cauchy–Schwarz inequality in the framework of semi-inner product C^*-modules over unital C^*–algebras and applied it to present a Kantorovich type inequality.

In this paper we present some operator Kantorovich inequalities involving unital positive linear mappings and the operator geometric mean in the framework of semi-inner product C^*-modules and get some new and classical results in a unified approach.

2. A Kantorovich inequality via operator geometric mean

We provide a generalization of the Kantorovich inequality in the context of Hilbert C^*-modules which can be viewed as an extension of inequality (1.6) of Nakamoto and Nakamura [32].

Let $x, y, z, x_1, \ldots, x_n$ be arbitrary elements of a semi-inner product $\mathscr{A}$-module $(\mathscr{X}, \langle\cdot,\cdot\rangle)$. The authors of [4] studied the covariance $\mathrm{cov}_z(x,y) := \| z \|^2 \langle x,y\rangle - \langle x,z\rangle \langle z,y\rangle$ and the variance $\mathrm{var}_z(x) = \mathrm{cov}_z(x,x)$, and proved that $[\mathrm{cov}_z(x_i,x_j)] \in \mathbb{M}_n(\mathscr{A})$ is positive, or equivalently

$$\| z \|^2 \left[\langle x_i, x_j\rangle \right] \geq \left[\langle x_i, z\rangle \langle z, x_j\rangle \right] \quad \text{(Generalized Covariance-Variance Inequality)}. \tag{2.1}$$

In particular, by the Cauchy–Schwarz inequality for the semi-inner product $\mathrm{cov}_z(.,.)$, it holds

$$\mathrm{cov}_z(x,y)\,\mathrm{cov}_z(x,y)^* \leq \| \mathrm{var}_z(y) \| \,\mathrm{var}_z(x) \qquad \text{(Covariance-Variance Inequality)}.$$

Recall that for positive invertible elements $a, b \in \mathscr{A}$, we can use the following characterization of operator mean due to Ando as follows

$$a\sharp b = \max\left\{x \in \mathscr{A} : x = x^*, \begin{bmatrix} a & x \\ x & b \end{bmatrix} \geq 0\right\},$$

where $a\sharp b = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$. This is easily deduced from $a = (a\sharp b)b^{-1}(a\sharp b)$ and the fact that $a \geq xb^{-1}x^*$ if and only if $\begin{bmatrix} a & x \\ x^* & b \end{bmatrix} \geq 0$, where $x \in \mathscr{A}$.

Theorem 2.1. *Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras, $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module and $A \in \mathbb{B}(\mathscr{X})$ such that $0 < m \leq A \leq M$ for some scalars m, M. Then for every $x \in \mathscr{X}$ for which $\langle x, x\rangle$ is invertible and every unital positive linear mapping $\Phi : \mathscr{A} \to \mathscr{B}$ it holds*

$$\Phi(\langle x, x\rangle) \leq \Phi(\langle Ax, x\rangle)\sharp\Phi(\langle A^{-1}x, x\rangle) \leq \frac{M+m}{2\sqrt{mM}}\Phi(\langle x, x\rangle). \tag{2.2}$$

Proof. First note that due to invertibility of $\langle x, x\rangle$ we have, for $a := mM\langle A^{-1}x, x\rangle$ and $b =: \langle Ax, x\rangle$, $a \geq m\langle x, x\rangle$ and $b \geq m\langle x, x\rangle$, so a and b are positive and invertible. Since Φ is positive and unital, $\Phi(a)$ and $\Phi(b)$ are also positive and invertible.

Observe now that $M - A$ and $\frac{1}{m} - A^{-1}$ are positive commuting elements of the C^*-algebra $\mathbb{B}(\mathscr{X})$. This implies that $(M - A)(\frac{1}{m} - A^{-1}) \geq 0$, wherefrom we get $mMA^{-1} + A \leq (m + M)$. Then for every $x \in \mathscr{X}$

$$mM\langle A^{-1}x, x\rangle + \langle Ax, x\rangle \leq (m + M)\langle x, x\rangle$$

and therefore

$$mM\Phi(\langle A^{-1}x, x\rangle) + \Phi(\langle Ax, x\rangle) \leq (m + M)\Phi(\langle x, x\rangle). \tag{2.3}$$

Since $\sqrt{t} \leq \frac{1+t}{2}$ $(t \geq 0)$, for any $a, b \in \mathscr{A}$ we get

$$(\Phi(a)^{-\frac{1}{2}}\Phi(b)\Phi(a)^{-\frac{1}{2}})^{\frac{1}{2}} \leq \frac{1}{2}(e + \Phi(a)^{-\frac{1}{2}}\Phi(b)\Phi(a)^{-\frac{1}{2}})$$

and then $\Phi(a)^{\frac{1}{2}}(\Phi(a)^{-\frac{1}{2}}\Phi(b)\Phi(a)^{-\frac{1}{2}})^{\frac{1}{2}}\Phi(a)^{\frac{1}{2}} \leq \frac{1}{2}(\Phi(a)+\Phi(b))$. Hence $\Phi(a)\sharp\Phi(b) \leq \frac{1}{2}(\Phi(a) + \Phi(b))$. Thus we get

$$\begin{aligned}\sqrt{mM}\Phi(\langle A^{-1}x, x\rangle)\sharp\Phi(\langle Ax, x\rangle) &\leq \frac{1}{2}(mM\Phi(\langle A^{-1}x, x\rangle) + \Phi(\langle Ax, x\rangle)) \\ &\leq \frac{m+M}{2}\Phi(\langle x, x\rangle),\end{aligned}$$

which gives the Kantorovich inequality (the second inequality of (2.2)).

Applying (2.1) for $n = 2$, $x_1 = A^{\frac{1}{2}}x, x_2 = A^{-\frac{1}{2}}x$ and an arbitrary z such that $z \neq 0$ we get

$$\begin{bmatrix} \langle Ax, x\rangle & \langle x, x\rangle \\ \langle x, x\rangle & \langle A^{-1}x, x\rangle \end{bmatrix} \geq 0.$$

Now from [6, Corollary 4.4 (ii)] it follows that

$$\begin{bmatrix} \Phi(\langle Ax, x\rangle) & \Phi(\langle x, x\rangle) \\ \Phi(\langle x, x\rangle) & \Phi(\langle A^{-1}x, x\rangle) \end{bmatrix} \geq 0,$$

so $\Phi(\langle x, x\rangle) \leq \Phi(\langle Ax, x\rangle)\sharp\Phi(\langle A^{-1}x, x\rangle)$. □

Corollary 2.2. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a unital positive linear map. If $A \in \mathbb{B}(\mathscr{H})$ be an operator satisfying $0 < m \leq A \leq M$ for some scalars m, M. Then*

$$\Phi(A)\sharp\Phi(B) \leq \frac{M+m}{2\sqrt{Mm}}. \tag{2.4}$$

Proof. Consider $\mathscr{X} = \mathbb{B}(\mathscr{H})$ regarded as a $\mathbb{B}(\mathscr{H})$-Hilbert module under $\langle T, S\rangle = T^*S$. Then $\mathbb{B}(\mathscr{X}) = \mathbb{B}(\mathscr{H})$ and if we take x to be the identity operator on $\mathscr{H}$, then (2.4) follows from (2.2). □

3. Some Kantorovich type inequalities

We present an additive version of the Kantorovich inequality.

Theorem 3.1. *Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras, $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module and $A \in \mathbb{B}(\mathscr{X})$ such that $0 < m \leq A \leq M$ for some scalars m, M. Then for every $x \in \mathscr{X}$ for which $|x| = e$, where e denotes the unit of $\mathscr{A}$, and every unital positive linear mapping $\Phi : \mathscr{A} \to \mathscr{B}$ it holds*

$$\Phi(\langle A^{-1}x, x\rangle) - \Phi(\langle Ax, x\rangle)^{-1} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{Mm}.$$

Proof. As Φ is unital it follows from (2.3) that

$$\Phi(\langle A^{-1}x, x\rangle) \leq \frac{M+m}{Mm} - \frac{1}{Mm}\Phi(\langle Ax, x\rangle). \tag{3.1}$$

Hence

$$\begin{aligned}
\Phi(\langle A^{-1}x,x\rangle) &- \Phi(\langle Ax,x\rangle) \\
&\leq \frac{M+m}{Mm} - \frac{1}{Mm}\Phi(\langle Ax,x\rangle) - \Phi(\langle Ax,x\rangle) \\
&= \frac{M+m}{Mm} - \frac{1}{Mm}\Phi(\langle Ax,x\rangle) - \Phi(\langle Ax,x\rangle) \\
&= \frac{M+m}{Mm} - \left(\frac{1}{\sqrt{Mm}}\Phi(\langle Ax,x\rangle)^{1/2} - \Phi(\langle Ax,x\rangle)^{1/2}\right)^2 - \frac{2}{\sqrt{Mm}} \\
&= \frac{(\sqrt{M}-\sqrt{m})^2}{Mm} - \left(\frac{1}{\sqrt{Mm}}\Phi(\langle Ax,x\rangle)^{1/2} - \Phi(\langle Ax,x\rangle)^{1/2}\right)^2 \\
&\leq \frac{(\sqrt{M}-\sqrt{m})^2}{Mm}.
\end{aligned}$$

□

There is still another multiplication type of the Kantorovich inequality as follows.

Theorem 3.2. *Under the conditions as in Theorem 3.1 it holds*

$$\Phi(\langle A^{-1}x,x\rangle) \leq \frac{(M+m)^2}{4Mm}\Phi(\langle Ax,x\rangle)^{-1}.$$

Proof. It follows from (3.1) that

$$\begin{aligned}
\Phi(\langle A^{-1}x,x\rangle) &\leq \frac{(M+m)^2}{4Mm}\left(\frac{4}{M+m} - \frac{4}{(M+m)^2}\Phi(\langle Ax,x\rangle)\right) \\
&\leq \frac{(M+m)^2}{4Mm}\Phi(\langle Ax,x\rangle)^{-1}
\end{aligned}$$

since $2\alpha\beta - \alpha^2 \leq \beta^2$ for real numbers. □

Finally we present a noncommutative version of (1.3).

Theorem 3.3. *Let $\mathscr{A}$ be a unital C^*-algebra with unit e, $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module and $A \in \mathbb{B}(\mathscr{X})$ such that $0 < m \leq A \leq M$ for some scalars m, M. Then for every $x \in \mathscr{X}$*

$$\langle Ax,x\rangle \leq \langle A^2x,x\rangle \,\sharp\, \langle x,x\rangle \leq \frac{M+m}{2\sqrt{mM}}\langle Ax,x\rangle \quad (x \in \mathscr{X}).$$

In particular, if $\langle x,x\rangle = e$ and $\mathscr{A}$ is commutative, then

$$\langle Ax,x\rangle^2 \le \langle A^2x,x\rangle \le \frac{M+m}{2\sqrt{mM}}\langle Ax,x\rangle^2. \tag{3.2}$$

Proof. Let $x \in \mathscr{X}$ be arbitrary. We have

$$\left\langle A^{1/2}x|A^{1/2}x|^{-1}, A^{1/2}x|A^{1/2}x|^{-1}\right\rangle = |A^{1/2}x|^{-1}\left\langle A^{1/2}x, A^{1/2}x\right\rangle |A^{1/2}x|^{-1} = e$$

Replacing x by $A^{1/2}x|A^{1/2}x|^{-1}$ in Theorem 2.1 we get

$$\begin{aligned}
&\langle A^{1/2}x|A^{1/2}x|^{-1}, A^{1/2}x|A^{1/2}x|^{-1}\rangle\\
&\le \langle A(A^{1/2}|A^{1/2}x|^{-1}), A^{1/2}|A^{1/2}x|^{-1}\rangle \sharp \langle A^{-1}(A^{1/2}x|A^{1/2}x|^{-1}), A^{1/2}x|A^{1/2}x|^{-1}\rangle\\
&\qquad\le \frac{M+m}{2\sqrt{mM}}\left\langle A^{1/2}x|A^{1/2}x|^{-1}, A^{1/2}x|A^{1/2}x|^{-1}\right\rangle,
\end{aligned}$$

whence

$$\begin{aligned}
&|A^{1/2}x|^{-1}\langle A^{1/2}x, A^{1/2}x\rangle|A^{1/2}x|^{-1}\\
&\le \left(|A^{1/2}x|^{-1}\langle A^2x,x\rangle|A^{1/2}x|^{-1}\right)\sharp\left(|A^{1/2}x|^{-1}\langle x,x\rangle|A^{1/2}x|^{-1}\right)\\
&\qquad\le \frac{M+m}{2\sqrt{mM}}|A^{1/2}x|^{-1}\left\langle A^{1/2}x, A^{1/2}x\right\rangle|A^{1/2}x|^{-1}.
\end{aligned}$$

Using the property $(c^*ac)\sharp(c^*bc) = c(a\sharp b)c$ of the operator geometric mean, we get

$$\langle Ax,x\rangle \le \langle A^2x,x\rangle \sharp \langle x,x\rangle \le \frac{M+m}{2\sqrt{mM}}\langle Ax,x\rangle.$$

The special case follows from the property $a\sharp e = a^{1/2}$ and the fact that a C^*-algebra is commutative if and only if $a \le b \Rightarrow a^2 \le b^2$ for all its elements a, b. □

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Department of Pure Mathematics, Ferdowsi University of Mashhad, Center of Excellence in Analysis on Algebraic Structures (CEAAS), P. O. Box 1159, Mashhad 91775, Iran.

E-mail address: moslehian@um.ac.ir, moslehian@member.ams.org

URL: http://profsite.um.ac.ir/~moslehian/